Transformations of Simple Polynomial Functions

Exercises

1. In an appropriate order,

   a. Describe the transformations that must be applied to $y = x^3$ to obtain the graph of $y = \left[-3(x+1)\right]^3 - 2$.

   b. Describe the transformations that must be applied to $y = x^4$ to obtain the graph of $y = -\frac{1}{5}(x-4)^4 + 1$.

2. Match each graph with its corresponding function.

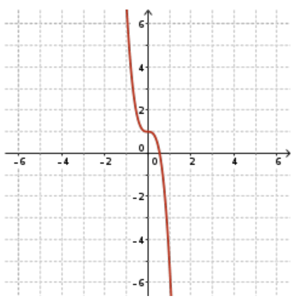
   a. $y = 7(x-2)^2 - 1$

   b. $y = -6x^3 + 1$

   c. $y = (x-4)^5 + 2$

   d. $y = (x-2)^4 - 5$

I.



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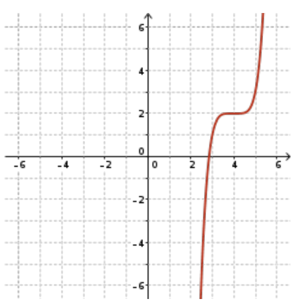

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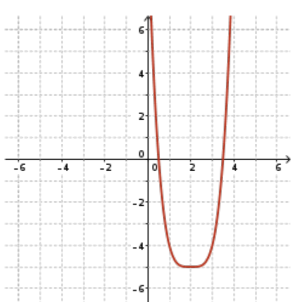

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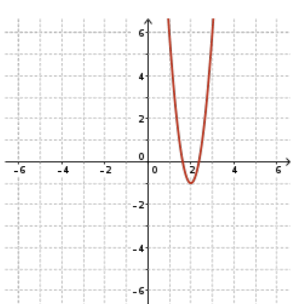


3. For each function, $y = f(x)$,

a. Identify the parent function and describe the transformations that must be applied to the parent function to obtain the graph of $y = f(x)$.

b. Graph $y = f(x)$ and the parent function on the same set of axes. State the domain and range of $y = f(x)$.

i. $f(x) = -2(x+3)^3$

ii. $f(x) = \frac{1}{2}x^4 - 5$

iii. $f(x) = \left(\frac{1}{2}x - 1\right)^3$

iv. $f(x) = -(2x)^4 + 7$

4. Describe, using mapping notation, the image of an arbitrary point (x, y) on the graph of $y = x^3$ under the following transformations and state the equation of the transformed function.

a. Vertical stretch about the x-axis by a factor of 3, vertical translatation 2 units up

b. Horizontal stretch about the y-axis by a factor of 4, horizontal translation left 2 units

c. Reflection in the y-axis, vertical stretch about the x-axis by a factor of $\frac{1}{2}$, horizontal translation right 3 units

d. Reflection in the x-axis, horizontal stretch about the y-axis by a factor of $\frac{1}{4}$, vertical translation 2 units up

5. Express each equation in the simplified form $y = a(x-h)^n + k$.

a. $y = \frac{2}{9}(-3x+6)^3 + 4$

b. $y = -48(-\frac{1}{2}x + 2)^4 + 2$

6. Determine an equation for the transformed function obtained from each set of transformations described below. Express the equation in the simplified form $y = a(x-h)^n + k$.

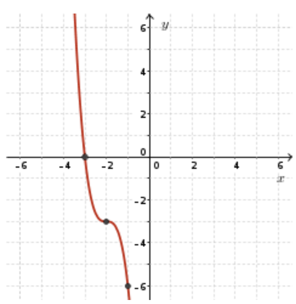
a. The function $f(x) = x^3$ vertically stretched about the x-axis by a factor of 2, and translated downwards by 5 units.

b. The function $f(x) = x^4$ reflected in the x-axis, reflected in the y-axis, vertically stretched about the x-axis by a factor of $\frac{1}{2}$, horizontally stretched about the y-axis by a factor of $\frac{3}{2}$, translated right 2 units, and down 3 units..

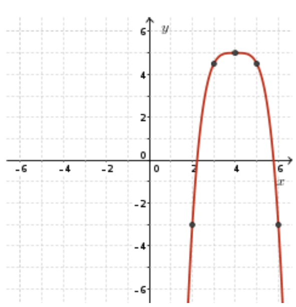
c. The function $f(x) = (x-3)^4 + 1$ reflected in the y-axis, translated 2 units left, and 1 unit up.

7. For the following graphs, determine an equation for the transformed function in the form $y = a(x-h)^n + k$.

a.



b.



8. Given the point $P(x, y)$ on the base graph of the form $y = x^n, n \in \mathbb{N}$, identify the corresponding point on the transformed function using mapping notation.

a. $y = (x-2)^3 + 4$

b. $y = \frac{5}{2}x^2 - 7$

c. $y = 2(x + \frac{5}{2})^4 - 1$

d. $y = -(-2x+5)^5 + 8$

9. Given a point $P(x, y)$ on the following transformed functions, describe (using mapping notation) the corresponding point on the base graph of

the form $y = x^n, n \in \mathbb{N}$, that maps to P.

a. $y = (x+2)^2 - 1$

b. $y = \frac{1}{5}x^4$

c. $y = -(2x-4)^3$

d. $y = -\frac{1}{2}(x-3)^5 + 4$

10. How many x-intercepts will the function

$$f(x) = -(x+2)^n - 2, n \in \mathbb{N}$$

have if

a. $n = 3$?

b. $n = 4$?

c. $n = 2k+1, k \in \mathbb{N}$?

d. $n = 2k, k \in \mathbb{N}$?

11. a. The function $h(x) = 2(x-4)(x+2)(x-3)$ is reflected in the x-axis, vertically stretched about the x-axis by a factor of $\frac{5}{2}$, and translated 4 units left, 5 units down. Write an equation for the transformed function.

b. What transformations are applied to the function $p(x) = 3(2x-4)(x+2)(x-3)$ to obtain the function $q(x) = (x-2)(x-1)(x+3)$?

12. The graph of the function $f(x) = (bx)^n + k, n \in \mathbb{N}$ is vertically stretched about the x-axis by a factor of 2, reflected in the x-axis, horizontally stretched about the y-axis by a factor of 3, translated right 5 units, then down 1 unit. Three points on this new graph $g(x)$ are $(5, -21)$, $(14, -75)$, $(11, -37)$. Determine the equation of $y = f(x)$ and the equation of the new function $g(x)$ under the transformations applied.

Transformations of Simple Polynomial Functions

Partial Solutions

1. There is no solution provided for this question.

2. a. The parent function is $y = x^2$. From the equation, the parent function is stretched vertically about the x-axis by a factor of 7, translated 2 units right, and 1 unit down. This means it matches the graph of IV.
   b. The parent function is $y = x^3$. From the equation, the parent function is reflected in the x-axis, stretched vertically about the x-axis by a factor of 6, and translated 1 unit up. This means it matches the graph of I.
   c. The parent function is $y = x^5$. From the equation, the parent function is translated 4 units right and 2 units up. This means it matches the graph of II.
   d. The parent function is $y = x^4$. From the equation, the parent function is translated 2 units right and 5 units down. This means it matches the graph of III.

3. There is no solution provided for this question.

4. a. A vertical stretch about the x-axis by a factor of 3 is accomplished by multiplying the y-coordinate by 3 ($a = 3$). A vertical translation 2 units up is then attained by adding 2 to the y-coordinate ($k = 2$).
   $$\therefore (x, y) \to (x, 3y + 2) \text{ and } y = 3x^3 + 2$$
   b. A horizontal stretch about the y-axis by a factor of 4 is accomplished by multiplying the x-coordinate by 4 ($b = \frac{1}{4}$). A horizontal translation 2 units left is accomplished by subtracting 2 from the x-coordinate ($h = -2$).
   $$\therefore (x, y) \to (4x - 2, y) \text{ and } y = \left(\frac{1}{4}(x + 2)\right)^3$$
   c. A reflection in the y-axis is attained by multiplying the x-coordinate by -1 ($b = -1$). A vertical stretch about the x-axis by a factor of $\frac{1}{2}$ is accomplished by multiplying the y-coordinate by $\frac{1}{2}$ ($a = \frac{1}{2}$). A horizontal translation 3 units right is then achieved by adding 3 to the x-coordinate $(h = 3)$.
   $$\therefore (x, y) \to (-x + 3, \frac{1}{2}y) \text{ and } y = \frac{1}{2}(-(x - 3))^3$$
   d. A reflection in the x-axis is obtained by multiplying the y-coordinate by -1 ($a = -1$). A horizontal stretch about the y-axis by a factor of $\frac{1}{4}$ is accomplished by multiplying the x-coordinate by $\frac{1}{4}$ ($b = 4$). A vertical translation 2 units up is then attained by adding 2 to the y-coordinate ($k = 2$).
   $$\therefore (x, y) \to (\frac{1}{4}x, -y + 2) \text{ and } y = -(4x)^3 + 2$$

5. There is no solution provided for this question.

6. a. Representing the described transformations using function notation, the transformed function is
   $$\begin{aligned} y &= 2f(x) - 5 \\ &= 2\left(x^3\right) - 5 \\ &= 2x^3 - 5 \end{aligned}$$
   b. Representing the described transformations using function notation, the transformed function is
   $$\begin{aligned} y &= -\frac{1}{2}f\left(-\frac{2}{3}(x - 2)\right) - 3 \\ &= -\frac{1}{2}\left(-\frac{2}{3}(x - 2)\right)^4 - 3 \\ &= -\frac{8}{81}(x - 2)^4 - 3 \end{aligned}$$
   c. Representing the described transformations using function notation, the transformed function is
   $$\begin{aligned} y &= f(-(x + 2)) + 1 \\ &= \left[(-(x + 2) - 3)^4 + 1\right] + 1 \\ &= (-(x + 5))^4 + 2 \\ &= (x + 5)^4 + 2 \end{aligned}$$

7. There is no solution provided for this question.

8. a. From the equation of the transformed function, a point (x, y) is translated right 2 units and up 4 units. This corresponds to the mapping notation
   $$(x, y) \to (x + 2, y + 4)$$
   b. From the equation of the transformed function, a point (x, y) is stretched vertically about the x-axis by a factor of $\frac{5}{2}$ and translated down 7 units. This corresponds to the mapping notation
   $$(x, y) \to (x, \frac{5}{2}y - 7)$$
   c. From the equation of the transformed function, a point (x, y) is stretched vertically about the x-axis by a factor of 2, translated left $\frac{5}{2}$ units, and translated down 1 units. This corresponds to the mapping notation

$$(x, y) \to (x - \frac{5}{2}, 2y - 1)$$

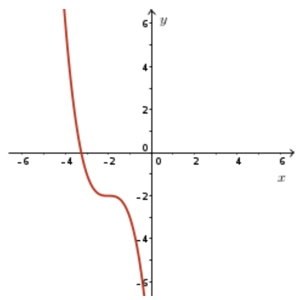
d. Factoring the terms within the parentheses gives $y = -\left[-2\left(x - \frac{5}{2}\right)\right]^5 + 8$. From the equation of the transformed function, a point (x, y) is reflected in the x-axis, reflected in the y-axis, stretched horizontally about the y-axis by a factor of $\frac{1}{2}$, translated right $\frac{5}{2}$ units, and translated up 8 units. This corresponds to the mapping notation

$$(x, y) \to (-\frac{1}{2}x + \frac{5}{2}, -y + 8)$$

9. There is no solution provided for this question.

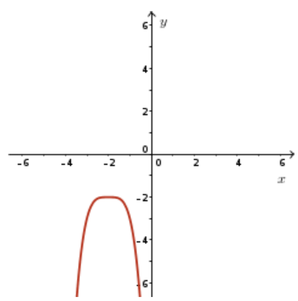
10. a.

If $n = 3$ then the function is a transformation of $y = x^3$ with the end behaviours of the graph heading in opposite directions. Therefore the graph of this function will cross the x-axis once.



b.

If $n = 4$ then the function is a transformation of $y = x^4$. The function opens downward (reflection in the x-axis) with end behaviours of the graph in the same direction, and has a turning point at $(-2, -2)$ (translation left 2 and down 2). Therefore the function has no x-intercept.



c. If $n = 2k + 1, k \in \mathbb{N}$ then the function will be a transformation of the power function $y = x^n$, where n is an odd integer greater than or equal to 3. Such functions have a shape the same or similar to $y = x^3$ with no turning points and the end behaviours of the graph heading in opposite directions. Therefore the graph of this function will cross the x-axis once.

d. If $n = 2k, k \in \mathbb{N}$ then the function will be a transformation of the power function $y = x^n$, where n is an even integer greater than or equal to 2. Such functions have a shape the same or similar to $y = x^2$ with a turning point and the end behaviours of the graph heading in the same direction. Since the turning point is at $(-2, -2)$ and the graph opens downward, the function will have no x-intercept.

11. There is no solution provided for this question.

12. The transformations that map points on $f(x)$ to points on $g(x)$ can be defined by

$$(x, y) \to (3x + 5, -2y - 1)$$

To determine the point (x, y) on $f(x)$ that maps to $(5, -21)$ on $g(x)$,

$$5 = 3x + 5 \text{ and } -21 = 2y - 1$$
$$0 = x \text{ and } 10 = y$$

Therefore, $(0, 10)$ is the corresponding point on $f(x)$.

Similarly, for $(14, -75)$ on $g(x)$

$$14 = 3x + 5 \text{ and } -75 = 2y - 1$$
$$3 = x \text{ and } 37 = y$$

Therefore, $(3, 37)$ is the corresponding point on $f(x)$.

Finally, for $(11, -37)$ on $g(x)$

$$11 = 3x + 5 \text{ and } -37 = 2y - 1$$
$$2 = x \text{ and } 18 = y$$

Therefore, $(2, 18)$ is the corresponding point on $f(x)$.

Since $(0, 10)$ is a point on $f(x) = (bx)^n + k$,

$$10 = (b(0))^n + k$$
$$\therefore k = 10$$

Using $(3, 37)$, $(2, 18)$ and $k = 10$, we have

$$37 = (3b)^n + 10$$
$$(3b)^n = 27 \quad (1)$$

and

$$18 = (2b)^n + 10$$
$$(2b)^n = 8 \qquad (2)$$

Dividing (2) by (1) gives

$$\frac{(2b)^n}{(3b)^n} = \left(\frac{2}{3}\right)^n = \frac{8}{27}$$

so $n = 3$. Substituting this value into (1),

$$\begin{aligned}(3b)^3 &= 27\\ 3b &= 3\\ b &= 1\end{aligned}$$

Therefore, $f(x) = x^3 + 10$, and hence

$$\begin{aligned}g(x) &= -2f\left(\frac{1}{3}(x-5)\right) - 1\\ &= -2\left[\left(\frac{1}{3}(x-5)\right)^3 + 10\right] - 1\\ &= -2\left[\frac{1}{27}(x-5)^3\right] - 20 - 1\\ &= -\frac{2}{27}(x-5)^3 - 21\end{aligned}$$